

# EQUATIONS OF MOTION FOR TWO-BODY PROBLEM ACCORDING TO AN OBSERVER INSIDE THE GRAVITATIONAL FIELD

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ABSTRACT. In this paper we consider equations of motion for 2-body problem according to an observer close to one of the gravitational bodies. The influence of the Thomas precession of the observer's frame has an important role. The equations of motion are based on a nonlinear connection modified by the action of an orthogonal tensor which synchronizes the 4-velocities of the considered two bodies. Finally we present periastron shift according to an observer inside the gravitational field, using the orthonormal coordinates. This is different approach than that used in General Relativity where the periastron shift is only given as observed far from the massive bodies.

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## 1. INTRODUCTION

We present equations of motion based on a nonlinear connection representing a relation between the gravitational source and test particle. We use *ict* convention

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(see pp. 51 in [5] about *ct/ict* conventions). So, we work with the Euclidean metric  $\text{diag}(1, 1, 1, 1)$  and upper and lower indices will not differ. Indeed, we consider simply a Minkowski space, and shall use orthonormal coordinates.

In the recent paper [9] in flat Minkowski space it is obtained a general formula for frequency redshift/blueshift based on the 4-wave vector in flat Minkowski space, which simultaneously explains the Doppler effect, gravitational redshift and under one cosmological assumption it also explains the cosmological redshift and the blueshift arising from the Pioneer anomaly. For this reason, and the recent results about non-linear connection in Minkowski space [12], it is not necessary to employ the apparatus of curved spaces for gravitational research (like, for example, in General Relativity), so, we also choose to use the nonlinear connection in this paper. We will use only one gravitational potential and denote it by  $\mu = 1 + \frac{GM}{rc^2}$ , instead of the Newton gravitational potential  $\frac{GM}{r}$ .

We build the connection step by step, starting from simpler cases and then, by including additional relevant variables, we achieve its final form. The goal is to distinguish two general aspects: equations of motion for an observer far from the massive bodies and for an observer inside a gravitational field. The result is applied to periastron shift of binaries. If the observer is far from the massive bodies, our results are the same as in the General Relativity (GR).

For the equations of motion it is also important the position of the observer, i.e. whether he is away from the gravitational field, or inside the gravitational field. So, we can distinguish four cases:

1. The observer is far from the massive bodies and the coordinates are orthonormal;
2. The observer is inside the gravitational field and the coordinates are ordinary;
3. The observer is inside the gravitational field and the coordinates are orthonormal;
4. The observer is far from the massive bodies and the coordinates are ordinary.

Here orthonormal coordinates means coordinates of flat Minkowski space. In this paper, we focus mainly on the case 3. The first case is considered in our another work [12]. The case 4 (specially the Einstein-Infeld-Hoffmann equations) is considered in the GR. In this paper the similarities and differences among these four cases will be emphasized.

## 2. SOME PRELIMINARIES ABOUT A NON-LINEAR CONNECTION

Non-linear connection means that the property  $\nabla_{aX+bY} = a\nabla_X + b\nabla_Y$  is not satisfied. The reason is simply because the 4-vectors of velocities do not form a vector space and we will consider them as Lorentz boosts. The definition of the 4-velocities is made in such a way that they correspond to tangent vectors on geodesics on 4-dimensional manifold, but on the other side, it is clear that the special-relativistic addition does not support classical linear combinations. So, motions of objects in a gravitational field would be more accurately described with equations of motion using a nonlinear connection. The non-linear connection was introduced in recent paper [12] for observer far from the massive bodies (case 1.) and now we shall extend it for in order to study the case 3. We do that in three steps.

**2.1. Using an analogy from electromagnetism.** We will make a complete analogy with the electromagnetism, where instead of the charge  $e$  we will consider mass  $M$ , and instead of the potential  $\frac{e}{r}$  we will consider the gravitational potential  $\frac{GM}{r}$ , assuming that  $M$  has the same value in each inertial coordinate system. Further we will introduce an antisymmetric tensor  $\phi_{ij}$  analogous to the tensor of electromagnetic field. Let us consider the motion of a test body with mass  $m$  under a gravitational attraction of a body with mass  $M$ . The 4-vector of velocity of the gravitational body is denoted by

$$(U_1, U_2, U_3, U_4) = \frac{1}{\sqrt{1 - u^2/c^2}} \left( \frac{u_x}{ic}, \frac{u_y}{ic}, \frac{u_z}{ic}, 1 \right), \quad (2.1)$$

where  $\vec{u} = (u_x, u_y, u_z)$  is the corresponding 3-vector of velocity. The 4-vector of velocity of the test body with mass  $m$  is denoted by

$$(V_1, V_2, V_3, V_4) = \frac{1}{\sqrt{1 - v^2/c^2}} \left( \frac{v_x}{ic}, \frac{v_y}{ic}, \frac{v_z}{ic}, 1 \right). \quad (2.2)$$

At every point  $(x, y, z)$ , for a stationary gravitational body with point mass  $M$ , we define tensor  $\phi$  with

$$(\phi_{ij}) = \begin{bmatrix} 0 & 0 & 0 & \frac{GM}{r^3 c^2}(x - x_0) \\ 0 & 0 & 0 & \frac{GM}{r^3 c^2}(y - y_0) \\ 0 & 0 & 0 & \frac{GM}{r^3 c^2}(z - z_0) \\ -\frac{GM}{r^3 c^2}(x - x_0) & -\frac{GM}{r^3 c^2}(y - y_0) & -\frac{GM}{r^3 c^2}(z - z_0) & 0 \end{bmatrix}, \quad (2.3)$$

where  $(x_0, y_0, z_0)$  is the position of the gravitational body. So,  $c^2(\phi_{41}, \phi_{42}, \phi_{43})$  represents the Newton acceleration toward the gravitational body. Considering its placement in  $\phi_{ij}$ , it is analogous to the electric field  $\vec{E}$  in the electromagnetic tensor.

We can use Lorentz transformations for uniform motion of particles and also, the principle of superposition of weak fields, and now  $\phi_{ij}$  is determined if all gravitational bodies are moving uniformly.

For a gravitational body moving non-uniformly, we consider the components of the tensor  $\phi_{ij}$  at each space-time point as analogous to the electromagnetic tensor derived by the Lienard-Wiechert potentials. We assume that the gravitational interaction transmits with velocity  $c$ . So, we get the following analogous formulas as in electrodynamics

$$c^2(\phi_{41}, \phi_{42}, \phi_{43}) = -\frac{GM}{(R - \frac{\vec{R} \cdot \vec{u}}{c})^3} \left( \vec{R} - \frac{\vec{u}}{c} R \right) - \frac{GM}{c^2 (R - \frac{\vec{R} \cdot \vec{u}}{c})^3} \vec{R} \times \left[ \left( \vec{R} - \frac{\vec{u}}{c} R \right) \times \dot{\vec{u}} \right], \quad (2.4a)$$

$$\frac{c}{i}(\phi_{32}, \phi_{13}, \phi_{21}) = \frac{1}{R} \vec{R} \times (\phi_{41}, \phi_{42}, \phi_{43}). \quad (2.4b)$$

Here,  $\vec{u}$  is the velocity of the gravitational body,  $\vec{R}$  is the 3-vector from the gravitational body to the considered point  $(x, y, z, ict)$  in the chosen coordinate system calculated at the space-time point  $(x', y', z', ict')$  of the gravitational body, such that after time  $t - t'$  of transmission of the interaction, the effect arrives at the considered point  $(x, y, z, ict)$ . Thus,  $t'$  is solution of the equation

$$t = t' + \frac{R(t')}{c}. \quad (2.5)$$

In (2.4a)  $\dot{\vec{u}} = \partial \vec{u} / \partial t'$  and  $R = |\vec{R}|$ .

For uniform motion  $\dot{\vec{u}} = 0$ , so the equation (2.4a) reduces to

$$c^2(\phi_{41}, \phi_{42}, \phi_{43}) = -\frac{GM}{R^3} \vec{R} \frac{1 - \frac{u^2}{c^2}}{\left(1 - \frac{u^2}{c^2} \sin^2 \theta\right)^{3/2}}, \quad (2.6)$$

where  $\theta$  is the angle between  $\vec{R}$  and  $\vec{u}$ , and  $\vec{R}$  is the 3-vector from the gravitational body to the considered point at time  $t$ .

Since for 2-body problem  $\vec{R}$  is collinear with  $\vec{u}$ , the last term in (2.4a) can be neglected up to  $c^{-2}$ . So we shall use only (2.6).

Although we accepted some facts from the electromagnetism, we must emphasize that there are two essential differences, which will be considered in the subsections 2.2 and 2.3.

i) While the charge  $e$  in electrodynamics is invariant scalar in all coordinate systems, here the mass  $M$  is not invariant. The reason is that the mass depends on the velocity of the body, according to the Special Relativity. Thus, the tensor  $\phi$  must be modified.

ii) The Lorentz force acting on charged particles at the considered point depends on the electromagnetic field and does not depend on the velocity of the source of the electromagnetic field. Considering gravitational forces, the motion depends on the velocity of the source of gravitation.

**2.2. Role of the masses in gravitational force and in the tensor  $\phi$ .** A mass far from the massive bodies measured by an observer far from gravitational influence will be called proper mass and will be denoted by  $m$ ,  $M$ ,  $m_1$ ,  $m_2$ ,... An observer far from the massive bodies observing a body with proper mass  $m$  that has fallen into a gravitational field with potential  $\mu = 1 + \frac{GM}{Rc^2}$ , will measure the value  $\frac{m}{1 + \frac{GM}{Rc^2}}$  for the mass of the body. For a test body with small mass  $m$  with respect to the gravitational body, the mass  $\frac{m}{1 + \frac{GM}{Rc^2}} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$  will be unchanged up to  $c^{-2}$  during motion. This is in accordance with conservation of energy (potential + kinetic) in a gravitational field.

Let us consider two bodies with masses  $m_1$  and  $m_2$  on a distance  $R$  between their centers. Then, the gravitational force originating from the body with mass  $m_2$  which acts on the body with mass  $m_1$ , is assumed to be

$$\vec{f} = \frac{m_1}{1 + \frac{Gm_2}{Rc^2}} \nabla \frac{Gm_2}{R(1 + \frac{Gm_1}{Rc^2})}, \quad (2.7)$$

while the acceleration of the body with mass  $m_1$  is assumed to be

$$\vec{a} = \frac{1}{1 + \frac{Gm_2}{Rc^2}} \nabla \frac{Gm_2}{R(1 + \frac{Gm_1}{Rc^2})}. \quad (2.8)$$

Up to  $c^{-2}$ , the acceleration (2.8) can be written in the form

$$\vec{a} = -\frac{\vec{R}}{R} \frac{Gm_2}{R^2} \left( 1 - \frac{G(2m_1 + m_2)}{Rc^2} \right). \quad (2.9)$$

If we consider two bodies within a system of  $n$ -bodies, it is necessary to use a more general formula for the acceleration. The masses of two bodies with proper masses  $m_1$  and  $m_2$  in a gravitational field are

$$\frac{m_1}{(1 + \frac{Gm_2}{r_{12}c^2})(1 + \frac{Gm_3}{r_{13}c^2})(1 + \frac{Gm_4}{r_{14}c^2})\dots},$$

and

$$\frac{m_2}{(1 + \frac{Gm_1}{r_{12}c^2})(1 + \frac{Gm_3}{r_{23}c^2})(1 + \frac{Gm_4}{r_{24}c^2})\dots}$$

respectively, where  $r_{ij}$  is the distance between the bodies with masses  $m_i$  and  $m_j$ .

Now, analogously to (2.7) and (2.8), for the force/acceleration of the body with mass  $m_1$  we get

$$\vec{f} = \frac{m_1}{(1 + \frac{Gm_2}{r_{12}c^2})(1 + \frac{Gm_3}{r_{13}c^2})(1 + \frac{Gm_4}{r_{14}c^2})\dots} \nabla \frac{Gm_2}{r_{12}(1 + \frac{Gm_1}{r_{12}c^2})(1 + \frac{Gm_3}{r_{23}c^2})(1 + \frac{Gm_4}{r_{24}c^2})\dots}, \quad (2.10)$$

$$\vec{a} = \frac{1}{(1 + \frac{Gm_2}{r_{12}c^2})(1 + \frac{Gm_3}{r_{13}c^2})(1 + \frac{Gm_4}{r_{14}c^2})\dots} \nabla \frac{Gm_2}{r_{12}(1 + \frac{Gm_1}{r_{12}c^2})(1 + \frac{Gm_3}{r_{23}c^2})(1 + \frac{Gm_4}{r_{24}c^2})\dots}. \quad (2.11)$$

Analogously to (2.9) we obtain

$$\begin{aligned} \vec{a} = & \left[ \left( 1 + \frac{Gm_2}{r_{12}c^2} \right) \left( 1 + \frac{Gm_1}{r_{12}c^2} \right)^2 \left( 1 + \frac{Gm_3}{r_{13}c^2} \right) \left( 1 + \frac{Gm_3}{r_{23}c^2} \right) \right. \\ & \left. \cdot \left( 1 + \frac{Gm_4}{r_{14}c^2} \right) \left( 1 + \frac{Gm_4}{r_{24}c^2} \right) \dots \right]^{-1} \nabla \frac{Gm_2}{r_{12}}. \end{aligned} \quad (2.12)$$

The components of angular velocity are much smaller than the components of acceleration in  $\phi$ , so we can multiply the tensor  $\phi$  by the coefficient which stands in front of  $\nabla \frac{Gm_2}{r_{12}}$  in (2.12). This coefficient in (2.12) is a scalar in the Minkowskian space up to  $c^{-2}$ , and hence the product will preserve the tensor character of  $\phi$ .

For example, if  $m_1$  is a negligible small mass and  $m_2 = M$  is non-zero mass, then its acceleration is equal to

$$\vec{a} = -\frac{\vec{R}}{R} \frac{\frac{GM}{R^2}}{1 + \frac{GM}{Rc^2}}. \quad (2.13)$$

This acceleration can be written as

$$\vec{a} = c^2 \nabla \ln \left( 1 + \frac{GM}{Rc^2} \right). \quad (2.14)$$

Obviously, the potentials  $\mu = 1 + \frac{GM}{Rc^2}$  and  $C\mu$ , where  $C$  is a constant, lead to the same acceleration.

Now, let us consider the case when the observer is inside the gravitational field, close to the body with mass  $m_2$ . We want to determine the motion of the body with mass  $m_1$  with respect to the observer close to the body with mass  $m_2$ . The gravitational force originating from the body with mass  $m_2$  which acts on the body with mass  $m_1$ , now is given by

$$\vec{f} = \frac{m_1}{(1 + \frac{Gm_2}{Rc^2})(1 + \frac{Gm_1}{Rc^2})} \nabla \frac{Gm_2}{R} \quad (2.15)$$

instead of (2.7), while the corresponding acceleration of the body with mass  $m_1$  is given by

$$\vec{a} = \frac{1}{(1 + \frac{Gm_2}{Rc^2})(1 + \frac{Gm_1}{Rc^2})} \nabla \frac{Gm_2}{R}. \quad (2.16)$$

Namely, the observer sees that the other body has mass  $\frac{m_1}{(1 + \frac{Gm_2}{Rc^2})(1 + \frac{Gm_1}{Rc^2})}$ , because the potential

$$(1 + \frac{Gm_2}{Rc^2})(1 + \frac{Gm_1}{Rc^2}) \approx 1 + \frac{G(m_1 + m_2)}{Rc^2}$$

is such that its gradient yields the relative acceleration  $-\vec{R} \cdot \frac{G(m_1 + m_2)}{R^2}$  of the body with mass  $m_1$  with respect to him. Therefore, for the mass in (2.15) we used  $\frac{m_1}{(1 + \frac{Gm_2}{Rc^2})(1 + \frac{Gm_1}{Rc^2})}$ . The proper mass  $m_2$  is observed to be unchanged with respect to itself, and so in (2.15) the gradient is applied to  $\frac{Gm_2}{R}$ . Now, analogously to (2.9), the acceleration is

$$\vec{a} = -\frac{\vec{R}}{R} \frac{Gm_2}{R^2} \left( 1 - \frac{G(m_1 + m_2)}{Rc^2} \right). \quad (2.17)$$

**2.3. Influence of the velocity of the gravitational source in the equations of motion.** Notice that in a system of four orthonormal vectors  $A_{i1}$ ,  $A_{i2}$ ,  $A_{i3}$  and  $A_{i4}$ , where  $A_{i\alpha}$  is the  $i$ -th coordinate of the  $\alpha$ -th vector, using that  $A_{i\alpha}$  is an orthogonal matrix, the following tensor

$$\frac{dA_{i\alpha}}{ds} A_{j\alpha} \quad (2.18)$$

is also antisymmetric as  $\phi_{ij}$  is defined to be. In (2.18)  $ds = ic\sqrt{1 - \frac{v^2}{c^2}}dt$ , just like in the Special Relativity. For  $U_i = V_i$ , we identify  $\phi_{ij}$  with the tensor given in (2.18). Then, the physical interpretation of  $\phi_{ij}$  can be obtained using the tensor (2.18).

So,  $\phi$  is given by

$$\phi = \begin{bmatrix} 0 & -i\omega_z/c & i\omega_y/c & -a_x/c^2 \\ i\omega_z/c & 0 & -i\omega_x/c & -a_y/c^2 \\ -i\omega_y/c & i\omega_x/c & 0 & -a_z/c^2 \\ a_x/c^2 & a_y/c^2 & a_z/c^2 & 0 \end{bmatrix}, \quad (2.19)$$

where  $\vec{a} = (a_x, a_y, a_z)$  is the 3-vector of acceleration and  $\vec{w} = (w_x, w_y, w_z)$  is the 3-vector of angular velocity.

We will introduce an orthogonal tensor  $P(U, V)$ , which should have the properties: to be orthogonal and to transform the 4-velocity of the source into the 4-velocity of the particle. It can be understood as an apparatus for transition between the frames of the source and the particle. The tensor  $P = P(U, V)$  is given by

$$P_{ij} = \delta_{ij} - \frac{1}{1 + U_s V_s} (V_i V_j + V_i U_j + U_i V_j + U_i U_j) + 2U_j V_i. \quad (2.20)$$

We assume the equality

$$\frac{dA_{i\alpha}}{ds} A_{j\alpha} = P_{ri} \phi_{rk} P_{kj}, \quad (2.21)$$

which would represent a general formula for the parallel transport of the considered frame  $A_{i\alpha}$  in direction of the 4-vector of velocity  $V_i$ , or in matrix form

$$\frac{dA}{ds} A^T = P^T \phi P.$$

The tensor  $P_{ij}$  is an orthogonal matrix and it has the following property  $P(U, V) = P(V, U)^{-1}$ . Some other properties of this tensor and a justification for its appearance in (2.20) are given in [2, 8]. For example, it is shown that using the standard addition, one can not uniquely determine a 4-vector in the Minkowskian space-time which would represent a relative 4-velocity of a point  $B$  with respect to a point  $A$ , assuming that  $B$  moves with 4-velocity  $V$  and  $A$  moves with 4-velocity  $U$ . So, the tensor  $P(U, V)$  provides a transition between frames, i.e.  $P_{ij} U_j = V_i$ .



In the special case  $(U_i) = (0, 0, 0, 1)$ , the tensor  $P(U, V)$  is given by

$$P = \begin{bmatrix} 1 - \frac{1}{\nu}V_1^2 & -\frac{1}{\nu}V_1V_2 & -\frac{1}{\nu}V_1V_3 & V_1 \\ -\frac{1}{\nu}V_2V_1 & 1 - \frac{1}{\nu}V_2^2 & -\frac{1}{\nu}V_2V_3 & V_2 \\ -\frac{1}{\nu}V_3V_1 & -\frac{1}{\nu}V_3V_2 & 1 - \frac{1}{\nu}V_3^2 & V_3 \\ -V_1 & -V_2 & -V_3 & V_4 \end{bmatrix}, \quad (2.22)$$

where  $V_1, V_2, V_3, V_4$  are given by (2.2),  $\nu = 1 + V_4$ , and this represents just a Lorentz transformation (as a boost, without space rotation). If we multiply the equation (2.21) by  $A_{j\beta}$  and sum for  $\beta = 1, 2, 3, 4$  we get

$$\frac{dA_{i\beta}}{ds} = P_{ri}\phi_{rk}P_{kj}A_{j\beta},$$

and hence for the parallel displacement of arbitrary (unit) vector  $A_i$  we get

$$\frac{dA_i}{ds} = P_{ri}\phi_{rk}P_{kj}A_j. \quad (2.23)$$

Specially, for  $A_i = V_i$ , we obtain the equations for parallel displacement, i.e.

$$\frac{dV_i}{ds} = P_{ri}\phi_{rk}P_{kj}V_j, \quad (2.24)$$

and these are the equations of motion in orthonormal coordinates.

### 3. INFLUENCE OF THE THOMAS PRECESSION OF THE COORDINATE AXES

Let us return to the basic equations of motion (2.24). We mentioned that these equations are Lorentz covariant, but although they give the exact equations of motion, they are represented only *with respect to the chosen coordinates*, which is a usual practice, but it is not always the same with the observation with respect to the distant stars.

Let us consider a gravitational body, a test body with zero mass and a coordinate frame in a small neighborhood of the test body. In case of weak gravitational field we may assume that the gravitational field is caused by many small bodies (atoms) of the gravitational body, which have zero angular momentums. Observing from

the frame, the distant stars would make an apparent (not true) rotation on the sky with angular velocity

$$\vec{w} = \frac{1}{2} \sum_i \frac{(\vec{v} - \vec{u}_i) \times \vec{a}_i}{c^2} \quad (3.1)$$

where  $\vec{u}_i$  is the velocity of the  $i$ -th body,  $\vec{v}$  is the velocity of the observer from the chosen coordinate system and  $\vec{a}_i$  is the Newtonian acceleration of the test body toward the  $i$ -th gravitational body. This formula was obtained recently in [11] in order to obtain Lorentz covariance of the precession of the axis of a gyroscope. Notice that (3.1) does not depend on the choice of the coordinate system. This angular velocity of the distant stars on the sky is observed by the telescope from the Gravity Probe B experiment, but it is masked by much larger effects. Consequently, observed from a system far from the massive bodies, which rests with respect to the observer, i.e.  $\vec{v} = 0$ , the coordinate frame rotates with the opposite angular velocity

$$\vec{w}^* = \frac{1}{2} \sum_i \frac{\vec{u}_i \times \vec{a}_i}{c^2}. \quad (3.2)$$

Moreover, any coordinate system far from the massive bodies observes that the coordinate frame rotates with the same angular velocity (3.2) [11]. What does it mean that one observer sees that another coordinate system rotates with angular velocity  $w$  [11]? If we have two observers from coordinate systems  $S_1$  and  $S_2$ , and assume that the precession of a gyroscope's spin axis is observed to have angular velocity  $w_1$  and  $w_1 - w_2$  from  $S_1$  and  $S_2$  respectively, then we may define that the observer from  $S_1$  sees that the "coordinate system  $S_2$  rotates with angular velocity  $w_1 - w_2$ ".

The angular velocity (3.2) is analogous to the Thomas precession. While the Thomas precession is related with the gyroscopes, the angular velocity (3.2) is related to the coordinate frame near the massive bodies. We shall call this angular velocity also Thomas precession. This effect will be also applied in case of periastron shift in order to obtain the precession with respect to the distant stars. This is the main goal of this paper.

How this precession of frames influence the equations of motion? More precisely, what would be the modification/correction of the equations of motion, so they would give equations of motion with respect to a non-precessing coordinate system far from the massive bodies? In the equations of motion (2.24), all possible influences appearing in the chosen coordinate system are implicitly included. This

is a consequence of the covariance of these equations. The angular velocity (3.2) is a real angular velocity, and its influence should be subtracted from the tensor equation (2.24).

Now let us return to the required correction for the equations of motion of any particle with velocity  $\vec{v}$ . The periastron shift has to be Lorentz covariant, i.e. it should not depend on the observer if he is in an inertial coordinate system far from the massive bodies. For the considered particle we must also consider the Coriolis acceleration and the transverse acceleration, while the centrifugal acceleration which depends on  $w^2$  will be of order  $c^{-4}$  and can be neglected. Notice that the considered gravitational body should be considered as center of rotation. Thus for example for the Coriolis force we should take into account the relative velocity of the test particle with respect to the source of gravitation. We find the acceleration of the considered body caused by a particle with mass  $M_i$  and then all such accelerations should be summed. Hence, according to (3.2), the required correcting acceleration is

$$\vec{a}_{cor} = \sum_i \left( 2(\vec{v} - \vec{u}_i) \times \vec{w}_i^* + \vec{R}_i \times \frac{d\vec{w}_i^*}{dt} \right),$$

where  $\vec{w}_i^* = \frac{1}{2c^2}(\vec{u}_i \times \frac{GM_i(-\vec{R}_i)}{R_i^3})$ ,  $\vec{u}_i$  is the velocity of the  $i$ -th body,  $M_i$  is its mass,  $\vec{R}_i$  is the vector from the  $i$ -th body to the moving test body,  $R_i = |\vec{R}_i|$  and  $\vec{v}$  is the velocity of the observed coordinate system.

After some transformations using double vector products, we obtain

$$\begin{aligned} \vec{a}_{cor} = \frac{G}{c^2} \sum_i \left[ \frac{M_i}{2R_i^3} (\vec{v} - \vec{u}_i) (\vec{R}_i \cdot \vec{u}_i) + \frac{M_i}{R_i^3} \vec{R}_i \left( \vec{u}_i \cdot (\vec{v} - \vec{u}_i) \right) - \frac{3}{2} \frac{((\vec{v} - \vec{u}_i) \cdot \vec{R}_i)(\vec{u}_i \cdot \vec{R}_i)}{R_i^2} \right] - \\ - \frac{M_i}{2R_i} \dot{\vec{u}}_i + \frac{M_i}{2R_i^3} \vec{R}_i (\vec{R}_i \cdot \dot{\vec{u}}_i) \end{aligned} \quad (3.3)$$

#### 4. PERIASTRON SHIFT OF BINARY SYSTEMS

A straightforward calculation of the matrix  $S = P^T \phi P$ , where  $\phi$  is given by (2.19) and  $P$  is given by (2.22), leads to

$$\begin{aligned} S_{41} = -S_{14} &= i \frac{\omega_z}{c} V_2 - i \frac{\omega_y}{c} V_3 + \frac{a_x}{c^2} \left( V_4 + \frac{(V_1)^2}{1 + V_4} \right) + \frac{a_y}{c^2} \frac{V_1 V_2}{1 + V_4} + \frac{a_z}{c^2} \frac{V_1 V_3}{1 + V_4}, \\ S_{42} = -S_{24} &= i \frac{\omega_x}{c} V_3 - i \frac{\omega_z}{c} V_1 + \frac{a_x}{c^2} \frac{V_1 V_2}{1 + V_4} + \frac{a_y}{c^2} \left( V_4 + \frac{(V_2)^2}{1 + V_4} \right) + \frac{a_z}{c^2} \frac{V_2 V_3}{1 + V_4}, \\ S_{43} = -S_{34} &= i \frac{\omega_y}{c} V_1 - i \frac{\omega_x}{c} V_2 + \frac{a_x}{c^2} \frac{V_1 V_3}{1 + V_4} + \frac{a_y}{c^2} \frac{V_2 V_3}{1 + V_4} + \frac{a_z}{c^2} \left( V_4 + \frac{(V_3)^2}{1 + V_4} \right), \end{aligned}$$

$$\begin{aligned}
S_{32} &= -S_{23} = \frac{a_z}{c^2} V_2 - \frac{a_y}{c^2} V_3 + i \frac{\omega_x}{c} \left( V_4 + \frac{(V_1)^2}{1+V_4} \right) + i \frac{\omega_y}{c} \frac{V_1 V_2}{1+V_4} + i \frac{\omega_z}{c} \frac{V_1 V_3}{1+V_4}, \\
S_{13} &= -S_{31} = \frac{a_x}{c^2} V_3 - \frac{a_z}{c^2} V_1 + i \frac{\omega_x}{c} \frac{V_1 V_2}{1+V_4} + i \frac{\omega_y}{c} \left( V_4 + \frac{(V_2)^2}{1+V_4} \right) + i \frac{\omega_z}{c} \frac{V_2 V_3}{1+V_4}, \\
S_{21} &= -S_{12} = \frac{a_y}{c^2} V_1 - \frac{a_x}{c^2} V_2 + i \frac{\omega_x}{c} \frac{V_1 V_3}{1+V_4} + i \frac{\omega_y}{c} \frac{V_2 V_3}{1+V_4} + i \frac{\omega_z}{c} \left( V_4 + \frac{(V_3)^2}{1+V_4} \right), \\
S_{11} &= S_{22} = S_{33} = S_{44} = 0.
\end{aligned} \tag{4.1}$$

We will consider the periastron shift for a pulsar and its companion and denote by  $m$  the mass of a pulsar and by  $M$  the mass of its companion, assuming that both bodies are moving in the same plane ( $xy$ -plane). We choose a coordinate system such that its origin always passes through the line connecting the two bodies. Let  $(x, y, 0)$  be the coordinates of the pulsar, and let  $(x', y', 0)$  be the coordinates of its companion. We denote the 4-vector of velocity of the pulsar by

$$(V_i) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( \frac{v_x}{ic}, \frac{v_y}{ic}, \frac{v_z}{ic}, 1 \right), \tag{4.2}$$

and the 4-vector of velocity of its companion by

$$(U_i) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \left( \frac{u_x}{ic}, \frac{u_y}{ic}, \frac{u_z}{ic}, 1 \right). \tag{4.3}$$

It is convenient to use the notation  $R = \sqrt{(x - x')^2 + (y - y')^2}$ ,  $r = \sqrt{x^2 + y^2}$ ,  $\rho = 1/r$  and  $r \approx \frac{M}{M+m} R$ . If we make the replacements  $\frac{x - x'}{R} = \cos \alpha$  and  $\frac{y - y'}{R} = \sin \alpha$ , then  $\cos \alpha = \frac{x}{r}$ ,  $\sin \alpha = \frac{y}{r}$ ,  $x'/y' = x/y$ ,  $u_x \approx -v_x \frac{m}{M}$ , and  $u_y \approx -v_y \frac{m}{M}$ .

The basic equations of motion in orthonormal coordinates are

$$\frac{dV_i}{ds} = S_{ij} V_j,$$

where  $V_i$  is given by (4.2) and  $ds = ic \sqrt{1 - \frac{v^2}{c^2}}$ . These equations can be simplified into the following form

$$\frac{d^2 x}{dt^2} = -\sqrt{1 - \frac{v^2}{c^2}} (S_{41} v_x^2 + S_{42} v_x v_y + S_{43} v_x v_z) + ic \sqrt{1 - \frac{v^2}{c^2}} (S_{12} v_y + S_{13} v_z + ic S_{14}), \tag{4.4a}$$

$$\frac{d^2 y}{dt^2} = -\sqrt{1 - \frac{v^2}{c^2}} (S_{41} v_x v_y + S_{42} v_y^2 + S_{43} v_y v_z) + ic \sqrt{1 - \frac{v^2}{c^2}} (S_{21} v_x + S_{23} v_z + ic S_{24}), \tag{4.4b}$$

$$\frac{d^2 z}{dt^2} = -\sqrt{1 - \frac{v^2}{c^2}} (S_{41} v_x v_z + S_{42} v_y v_z + S_{43} v_z^2) + ic \sqrt{1 - \frac{v^2}{c^2}} (S_{31} v_x + S_{32} v_y + ic S_{34}), \tag{4.4c}$$

$$\frac{d}{dt} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = S_{41}v_x + S_{42}v_y + S_{43}v_z, \quad (4.4d)$$

where  $d^2x/dt^2 = dv_x/dt$ ,  $d^2y/dt^2 = dv_y/dt$  and  $d^2z/dt^2 = dv_z/dt$ . Specially for the 2-body problem in the  $xy$ -plane we can replace  $S_{34} = S_{13} = S_{23} = 0$ ,  $v_z = 0$  and we obtain the following simple system

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\sqrt{1 - \frac{v^2}{c^2}}(S_{41}v_x^2 + S_{42}v_xv_y) + ic\sqrt{1 - \frac{v^2}{c^2}}S_{12}v_y - S_{14}c^2\sqrt{1 - \frac{v^2}{c^2}}, \\ \frac{d^2y}{dt^2} &= -\sqrt{1 - \frac{v^2}{c^2}}(S_{41}v_xv_y + S_{42}v_y^2) - ic\sqrt{1 - \frac{v^2}{c^2}}S_{12}v_x - S_{24}c^2\sqrt{1 - \frac{v^2}{c^2}}. \end{aligned}$$

Up to  $c^{-2}$ , we get

$$\frac{d^2x}{dt^2} = -\left(1 - \frac{v^2}{2c^2}\right)c^2S_{14} - \frac{v_x}{c^2}(a_xv_x + a_yv_y) + icv_yS_{12}, \quad (4.5a)$$

$$\frac{d^2y}{dt^2} = -\left(1 - \frac{v^2}{2c^2}\right)c^2S_{24} - \frac{v_y}{c^2}(a_xv_x + a_yv_y) - icv_xS_{12}. \quad (4.5b)$$

Further, the components of the matrix  $S = P(U, V)^T \phi P(U, V)$  should be calculated analogously as in (4.1). In order to avoid large expressions with accuracy up to  $c^{-2}$ , it is sufficient to use the components (4.1), replacing  $v_x$  by  $v_x - u_x$  and  $v_y$  by  $v_y - u_y$ . Hence, for  $S_{14}$ ,  $S_{24}$ , and  $S_{12}$  we obtain

$$S_{14} = \left[ -a_x \left( \frac{1}{\sqrt{1 - \frac{(\vec{v} - \vec{u})^2}{c^2}}} - \frac{(v_x - u_x)^2}{2c^2} \right) + a_y \frac{(v_x - u_x)(v_y - u_y)}{2c^2} - w_z(v_y - u_y) \right] \frac{1}{c^2},$$

$$S_{24} = \left[ -a_y \left( \frac{1}{\sqrt{1 - \frac{(\vec{v} - \vec{u})^2}{c^2}}} - \frac{(v_y - u_y)^2}{2c^2} \right) + a_x \frac{(v_x - u_x)(v_y - u_y)}{2c^2} + w_z(v_x - u_x) \right] \frac{1}{c^2},$$

$$S_{12} = -\frac{i}{c} \left[ \frac{a_x}{c^2}(v_y - u_y) - \frac{a_y}{c^2}(v_x - u_x) + w_z \right].$$

According to (2.19), (2.6), (2.17), up to  $c^{-2}$ ,

$$\begin{aligned} \frac{1 - \frac{u^2}{c^2}}{(1 - \frac{u^2}{c^2} \sin^2 \theta)^{3/2}} &= \frac{1 - \frac{u^2}{c^2}}{(1 - \frac{u^2}{c^2})^{3/2} (1 + \frac{u^2}{c^2} \cos^2 \theta)^{3/2}} = \\ &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{1}{(1 + \frac{1}{c^2} (\vec{u} \cdot \frac{\vec{r}}{r})^2)^{3/2}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{1}{(1 + \frac{1}{c^2} \frac{m^2}{M^2} (\vec{r} \cdot \frac{\vec{r}}{r})^2)^{3/2}} = \\ &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{1}{(1 + \frac{1}{c^2} \frac{m^2}{M^2} (\frac{dr}{dt})^2)^{3/2}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{1}{(1 + \frac{1}{c^2} \frac{m^2}{(M+m)^2} (\frac{dR}{dt})^2)^{3/2}}, \end{aligned}$$

the components  $a_x$ ,  $a_y$ , and  $w_z$  are

$$\begin{aligned} a_x &= -\frac{x}{r} \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{GM}{R^2} \left(1 - \frac{G(M+m)}{Rc^2}\right) \lambda^{-3}, \\ a_y &= -\frac{y}{r} \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{GM}{R^2} \left(1 - \frac{G(M+m)}{Rc^2}\right) \lambda^{-3}, \\ w_z &= \frac{Gm}{R^2 c^2} \frac{v_x y - v_y x}{r} \lambda^{-3}, \end{aligned}$$

where  $\lambda = \sqrt{1 + \frac{1}{c^2} \frac{m^2}{(M+m)^2} \left(\frac{dR}{dt}\right)^2}$ . Further, we will omit this coefficient in expressions of type  $c^{-2}$ . Now, having the system of equations (4.5a) and (4.5b) for the motion of a body with mass  $m$  influenced by the gravitation of a body with mass  $M$ , we can calculate the periastron shift in two steps. The first step is consisted in finding an equation analogous to perihelion shift in orthonormal coordinates. In the second step we sum the equation (4.5a) multiplied by  $2v_x$  and the equation (4.5b) multiplied by  $2v_y$ . That equation can be integrated and the value of  $v^2$  can be found. After these two steps the periastron shift can be obtained. We present only the final results of these two steps avoiding the long algebraic and differential calculations.

We distinguish three aspects for obtaining the periastron shift and we number them by i), ii) and iii). In i) and ii) will be calculated the periastron shift with respect to the chosen coordinate system, using the non-linear connection from section 2. In iii) will be calculated periastron shift using the results from section 3.

i) We use the orthonormal coordinate system and we *assume a priori that the two bodies and the coordinate center are collinear*. In this case we can continue to deal with the system (4.5a,b) as previously described. Using the equalities between  $v_x$ ,  $u_x$ ;  $v_y$ ,  $u_y$ ;  $r$ ,  $R$  and so on, the system (4.5) can be reduced to the following form

$$\begin{aligned} \frac{d^2 \vec{r}}{dt^2} &= -\frac{\vec{R}}{R} \frac{GM}{R^2} \left[ 1 + \frac{V^2}{c^2} \frac{M^2 + 4Mm + 2m^2}{(M+m)^2} - \frac{G(M+m)}{Rc^2} - \right. \\ &\quad \left. - \frac{3}{2c^2} \frac{m^2}{(M+m)^2} \left(\frac{dR}{dt}\right)^2 \right] + \vec{V} \frac{dR}{dt} \frac{GM}{R^2} \left( \frac{M}{M+m} + \frac{3}{2} \right) \frac{1}{c^2}, \end{aligned} \quad (4.6)$$

where  $\vec{V}$  is the relative velocity of the pulsar with respect to its companion.

The first step ahead from the system (4.6) yields the following equation

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = v^2 C_2^{-2} \left[1 + \frac{5M + 3m}{M + m} \frac{GM\rho}{c^2}\right]. \quad (4.7)$$

The second step is more complicated. Although  $\lambda$  has a significant role in  $v^2$ ,  $\lambda$  has no influence in (4.7) and it has no role in the periastron shift [12]. So, we will omit  $\frac{3}{2c^2} \frac{m^2}{(M+m)^2} \left(\frac{dR}{dt}\right)^2$  in (4.6), i.e. we consider  $\lambda = 1$ . The second step yields

$$v^2 = 2 \frac{GM^3\rho}{(M+m)^2} - 4 \frac{M^6 G^2 \rho^2}{(M+m)^4 c^2} - 2 \frac{M^5 m G^2 \rho^2}{(M+m)^4 c^2} + \frac{C\rho}{c^2} + K, \quad (4.8)$$

where  $C$  and  $K$  are two mutually dependent constants, which do not play any role in the periastron shift. We get

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = A + B\rho + \frac{6G^2 M^4}{c^2 C_2^2 (M+m)^2} \cdot \left(1 + \frac{1}{3} \frac{Mm}{(M+m)^2}\right) \rho^2.$$

Since  $C_2 = \frac{2}{P} \pi a^2 \sqrt{1 - \epsilon^2} = \frac{2\pi}{P} a_r^2 \frac{M^2}{(M+m)^2} \sqrt{1 - \epsilon^2}$ , where  $a_r$  is the semi-major axis of the relative orbit and  $P$  is the orbital period, we obtain

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = A + B\rho + \frac{3P^2 G^2}{2\pi^2 c^2} \frac{(M+m)^2}{a_r^4 (1 - \epsilon^2)} \cdot \left(1 + \frac{1}{3} \frac{Mm}{(M+m)^2}\right) \rho^2.$$

So, for the periastron shift we obtain

$$\Delta\varphi = \frac{3G^2 (M+m)^2 P^2}{2\pi c^2 a_r^4 (1 - \epsilon^2)} \cdot \left(1 + \frac{1}{3} \frac{Mm}{(M+m)^2}\right).$$

Using that  $a_r^3 = P^2 G(M+m)/(4\pi^2)$ , we obtain finally

$$\Delta\varphi = \frac{6\pi(4\pi^2)^{1/3} G^{2/3}}{P^{2/3} c^2 (1 - \epsilon^2)} (M+m)^{2/3} \cdot \left(1 + \frac{1}{3} \frac{Mm}{(M+m)^2}\right). \quad (4.9)$$

ii) The equations of motion of the body with mass  $m$  are given by (4.6). Analogously to these equations, the equations of motion of the body with mass  $M$  (pulsar companion) are given by

$$\begin{aligned} \frac{d^2 \vec{r}}{dt^2} = & \frac{\vec{R}}{R} \frac{Gm}{R^2} \left[1 + \frac{V^2}{c^2} \frac{m^2 + 4Mm + 2M^2}{(M+m)^2} - \frac{G(M+m)}{Rc^2} - \right. \\ & \left. - \frac{3}{2c^2} \frac{M^2}{(M+m)^2} \left(\frac{dR}{dt}\right)^2\right] - \vec{V} \frac{dR}{dt} \frac{Gm}{R^2} \left(\frac{m}{M+m} + \frac{3}{2}\right) \frac{1}{c^2}. \end{aligned} \quad (4.10)$$

Subtracting the equation (4.10) from (4.6), after some transformations we get

$$\begin{aligned} \frac{d^2 \vec{R}}{dt^2} = & -\frac{\vec{R}}{R} \frac{G(M+m)}{R^2} \left[1 + \frac{V^2}{c^2} \frac{M^2 + 5Mm + m^2}{(M+m)^2} - \frac{G(M+m)}{Rc^2} - \right. \\ & \left. - \frac{3}{2c^2} \frac{Mm}{(M+m)^2} \left(\frac{dR}{dt}\right)^2\right] + \vec{V} \frac{dR}{dt} \frac{G}{R^2} \frac{5M^2 + 6Mm + 5m^2}{2(M+m)c^2}. \end{aligned} \quad (4.11)$$

While the assumption that the two bodies and the coordinate center are collinear was essential for deriving the equations (4.6) and (4.10), it has no role in the system (4.11), but (4.11) has a weakness because it is a subtraction between (4.6) and (4.10) which are deduced for different observers (coordinate systems). The equation (4.11) is independent of the coordinate system, it depends only on the relative parameters of the system.

A step ahead from the system (4.11), analogously to (4.7), yields the following equation

$$\left(\frac{d\frac{1}{R}}{d\varphi}\right)^2 + \frac{1}{R^2} = V^2 C_2^{-2} \left[1 + \frac{5M^2 + 6Mm + 5m^2}{M+m} \frac{G}{Rc^2}\right]. \quad (4.12)$$

Analogously to i) we can ignore the term  $\frac{3}{2c^2} \frac{Mm}{(M+m)^2} \left(\frac{dR}{dt}\right)^2$  in (4.11), because it has no role in the periastron shift and we obtain

$$V^2 = 2 \frac{G(M+m)}{R} - \frac{G^2}{R^2 c^2} (4M^2 - 2mM + 4m^2) + \frac{C}{Rc^2} + K, \quad (4.13)$$

where  $C$  and  $K$  are mutually dependent constants which have no role in the periastron shift. Hence, we come again to the same formula (4.9).

iii) Now, we will calculate the periastron shift in Lorentz invariant form where the Thomas precession from section 3 will be used. The results from section 3 must be used because (4.9) was deduced by assuming a priori that the two bodies and the coordinate origin are collinear. This includes an additional precession of the axis between the pulsar and the companion which should be subtracted from (4.9). The precession (4.9) should be corrected in the following way. Let  $O$  be an observer who rests with respect to the baricenter of the two bodies from an inertial coordinate system far from the massive bodies. According to (3.2),  $O$  observes the coordinate frame close to the pulsar that rotates with angular velocity

$$\frac{1}{2} \vec{u} \times \frac{-\vec{R}}{R^3} \frac{GM}{c^2},$$

while the coordinate frame close to the companion star which rotates, with angular velocity

$$\frac{1}{2} \vec{v} \times \frac{\vec{R}}{R^3} \frac{Gm}{c^2}.$$

Each of these two angular velocities can be written via the parameters of the relative orbit, i.e. each of them is equal to

$$\frac{1}{2} \vec{V} \times \frac{\vec{R}}{R^3} \frac{G(M+m)}{c^2} \cdot \frac{Mm}{(M+m)^2}, \quad (4.14)$$



where  $\vec{V}$  is the relative velocity of the pulsar with respect to its companion. So the line which connects the pulsar and its companion is observed to rotate additionally with angular velocity which is opposite of the sum of these two angular velocities, i.e. with

$$-\vec{V} \times \frac{\vec{R}}{R^3} \frac{G(M+m)}{c^2} \cdot \frac{Mm}{(M+m)^2}.$$

These two angular velocities do not change the distance from the pulsar to its companion, so they participate in the periastron shift observed via the coordinate system. The sum of both angular velocities should be integrated for a time of one orbital period. After some standard calculations, this yields the angle

$$\frac{1}{3} \frac{mM}{(M+m)^2} \frac{3G^2(M+m)^2 P^2}{2\pi c^2 a_r^4 (1-\epsilon^2)} = \frac{1}{3} \frac{mM}{(M+m)^2} \frac{6\pi(4\pi^2)^{1/3} G^{2/3}}{P^{2/3} c^2 (1-\epsilon^2)} (M+m)^{2/3}$$

per orbit. This angle is included in the total periastron shift (4.9) obtained via the covariant equations of motion, and so after its subtraction from (4.9) we obtain the precession

$$\Delta\varphi = \frac{6\pi(4\pi^2)^{1/3} G^{2/3}}{P^{2/3} c^2 (1-\epsilon^2)} (M+m)^{2/3}, \quad (4.15)$$

according to observer  $O$ . This formula (4.15) for the periastron shift is the same which predicts the GR, and it depends on the sum of the masses  $M+m$ .

If the system of two bodies moves with a constant velocity  $\vec{v}_0$ , then both velocities  $\vec{v}$  and  $\vec{u}$  should be replaced by  $\vec{v} + \vec{v}_0$  and  $\vec{u} + \vec{v}_0$ . In this case the moving observer observes the angular velocity

$$\begin{aligned} & -\frac{1}{2}(\vec{u} + \vec{v}_0) \times \frac{-\vec{R}}{R^3} \frac{GM}{c^2} - \frac{1}{2}(\vec{v} + \vec{v}_0) \times \frac{\vec{R}}{R^3} \frac{Gm}{c^2} = \\ & = -\vec{V} \times \frac{\vec{R}}{R^3} \frac{G(M+m)}{c^2} \cdot \frac{Mm}{(M+m)^2} + \frac{1}{2}\vec{v}_0 \times \left( \frac{\vec{R}}{R^3} \frac{GM}{c^2} - \frac{\vec{R}}{R^3} \frac{Gm}{c^2} \right). \end{aligned}$$

Thus, an additional anomalous angular velocity

$$\frac{1}{2}\vec{v}_0 \times \left( \frac{\vec{R}}{R^3} \frac{GM}{c^2} - \frac{\vec{R}}{R^3} \frac{Gm}{c^2} \right) \quad (4.16)$$

is observed. So the main problem is to consider the perturbations which come from (4.16).

First, notice that if  $M = m$ , then this angular velocity vanishes. In case of pulsar and its companion star, very often it is  $M \approx m$  and hence (4.16) is almost 0.

In general case, the constant vector  $\vec{v}_0$  can be decomposed as a sum of two vectors: a component which lies in the plane of rotation of the binary system, and

component which is orthogonal to that plane. We will consider these two special cases separately.

Assume that the vector  $\vec{v}_0$  lies in the orbital plane. Standard calculation shows that the integral of the vector (4.16) for a time of one period is zero. So, the total periastron shift remains unchanged with respect to the moving observer, i.e. it is given by (4.15) also.

Assume that the vector  $\vec{v}_0$  is orthogonal to the orbital plane. Then we have angular velocities in different planes, which are orthogonal to the orbital plane. We shall see now that the angular velocity (4.16) is a consequence from the Special Relativity. Let us denote by  $P$ ,  $C$ ,  $B$ , and  $O$  the pulsar, its companion, the barycenter and the moving observer respectively. For the sake of simplicity we assume that the eccentricity of the orbit is 0, i.e. the velocities of the pulsar and its companion are orthogonal to their radius-vectors. We denote the line  $OB$  as  $z$ -axis, and denote by  $\Sigma$  the plane through  $B$  which is orthogonal to  $OB$ . Now let us consider a composition of two Lorentz transformations: the first motion in the  $z$ -axis with velocity  $v_0$  (motion of the barycenter with respect to the observer), and motion with velocity  $v$  in a direction orthogonal to the  $z$ -axis (motion of the pulsar with respect to the barycenter). The composite Lorentz transformation shows that according to the observer  $O$ , the angle  $\angle OBP$  is not right, but there is a departure of  $\frac{v_0 v}{2c^2}$ . Hence the pulsar moves in a plane which is on distance  $rvv_0/(2c^2)$  from the plane  $\Sigma$ . Analogously, the companion star moves on a parallel plane which is on distance  $(R-r)vv_0/(2c^2)$  from the plane  $\Sigma$ . Notice that here  $v = |\vec{v}|$  and  $u = |\vec{u}|$ . Thus the angle between the axis  $CP$  and  $\Sigma$  is equal to

$$\begin{aligned} & \frac{1}{R} \left( \frac{rvv_0}{2c^2} - \frac{(R-r)uv_0}{2c^2} \right) = \frac{v_0}{2c^2} \left( \frac{r}{R}v - \frac{R-r}{R}u \right) = \\ & = \frac{v_0}{2c^2} \left( \frac{M^2}{(M+m)^2}V - \frac{m^2}{(M+m)^2}V \right) = \frac{1}{2} \frac{M-m}{M+m} \frac{v_0 V}{c^2}, \end{aligned}$$

where  $V$  is the relative velocity of the pulsar with respect to its companion. The last equality in vector form can be written as

$$-\frac{1}{2} \frac{M-m}{M+m} \frac{\vec{v}_0 \times \vec{V}}{c^2}.$$

Now if we differentiate this term by  $t$  we obtain (4.16).

Hence the conclusion that  $O$  observes the same trajectories of  $P$  and  $C$  but in different planes, and hence the periastron shift remains the same as (4.15) with

respect to the observer  $O$ . The moving observer sees that the orbital plane is not a fixed plane in the space. This consideration confirms the statement that *the "perturbations" arising from (4.16) and (3.2) have an important role to "cover" the equations up to Lorentz invariant equations of motion (2.24).*

Now we will obtain the same conclusion using the formula for acceleration (3.3). Using the equality  $\vec{v} : \vec{u} = -M : m$ , the corrected acceleration for the pulsar is

$$\vec{a}_P = \frac{G(M+m)}{2c^2 R^3} \vec{v}(\vec{R} \cdot \vec{u}) + \frac{G(M+m)}{c^2 R^3} \vec{R} \left( \vec{u} \cdot \vec{v} - \frac{3}{2} \frac{(\vec{v} \cdot \vec{R})(\vec{u} \cdot \vec{R})}{R^2} \right). \quad (4.17)$$

Symmetrically, for the corrected acceleration of the companion star we get

$$\vec{a}_C = \frac{G(M+m)}{2c^2 R^3} \vec{u}((- \vec{R}) \cdot \vec{v}) + \frac{G(M+m)}{c^2 R^3} (-\vec{R}) \left( \vec{u} \cdot \vec{v} - \frac{3}{2} \frac{(\vec{v} \cdot (-\vec{R}))(\vec{u} \cdot (-\vec{R}))}{R^2} \right),$$

i.e.

$$\vec{a}_C = -\vec{a}_P. \quad (4.18)$$

After these corrections of the accelerations of both bodies, the lines which pass through them will not intersect at the coordinate origin as previously. Thus, we can consider the periastron shift only via the relative orbit. The correction of the relative acceleration  $\vec{a}_R = \vec{a}_P - \vec{a}_C$  can be written in the form

$$\vec{a}_R = \left\{ \frac{G(M+m)}{c^2 R^3} \vec{v}(\vec{R} \cdot \vec{u}) + \frac{G(M+m)}{c^2 R^3} \vec{R} \left( \vec{u} \cdot \vec{v} - 3 \frac{(\vec{v} \cdot \vec{R})(\vec{u} \cdot \vec{R})}{R^2} \right) \right\} + \frac{G(M+m)}{c^2 R^3} \vec{R}(\vec{u} \cdot \vec{v}).$$

The first component

$$\frac{G(M+m)}{c^2 R^3} \vec{v}(\vec{R} \cdot \vec{u}) + \frac{G(M+m)}{c^2 R^3} \vec{R} \left( \vec{u} \cdot \vec{v} - 3 \frac{(\vec{v} \cdot \vec{R})(\vec{u} \cdot \vec{R})}{R^2} \right)$$

does not change the periastron shift, while the second component

$$\frac{G(M+m)}{c^2 R^3} \vec{R}(\vec{u} \cdot \vec{v}) \quad (4.19)$$

leads to the angle

$$\frac{1}{3} \frac{mM}{(M+m)^2} \frac{3G^2(M+m)^2 P^2}{2\pi c^2 a_r^4 (1-\epsilon^2)} = \frac{1}{3} \frac{mM}{(M+m)^2} \frac{6\pi(4\pi^2)^{1/3} G^{2/3}}{P^{2/3} c^2 (1-\epsilon^2)} (M+m)^{2/3}$$

per orbit. This angle is included in the total periastron shift (4.9), and after its subtraction from (4.9) we obtain (4.15).

**Remark.** If the observer is far from the massive bodies, the coordinate center coincides with the barycenter of the two bodies, while the periastron shift between the barycenter (coordinate origin) [12] and each of the two bodies is not the same with the periastron shift for the two bodies. In this paper, if the observer is one

of the bodies, ignoring the correction arising from the Thomas precession we have the opposite situation: the barycenter is not a fixed point and does not coincide with the coordinate origin at each moment, but the periastron shift between the coordinate origin and each of the bodies is the same as the periastron shift between the two bodies (cases i) and ii)).

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